Rainbow connectivity of the non-commuting graph of a finite group

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Abstract

Let G be a finite non-abelian group. The non-commuting graph Γ_G of G has the vertex set $G \setminus Z(G)$ and two distinct vertices x and y are adjacent if $xy \neq yx$, where $Z(G)$ is the center of G. We prove that the rainbow 2-connectivity of Γ_G is 2. In particular, the rainbow connection number of Γ_G is 2. Moreover, for any positive integer k, we prove that there exist infinitely many non-abelian groups G such that the rainbow k-connectivity of Γ_G is 2.

Key words: Non-commuting graph; non-abelian group; rainbow connectivity; rainbow path.

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1 Introduction

Let Γ be a connected graph with the vertex set $V(\Gamma)$ and the edge set $E(\Gamma)$. Given an edge coloring of Γ. A path P is *rainbow* if no two edges of P are colored the same. The *vertex connectivity* of Γ, denoted by $\kappa(\Gamma)$, is the smallest number of vertices whose deletion from Γ disconnects it. For any positive integer $k \leq \kappa(\Gamma)$, an edge-colored graph is called *rainbow-k-connected* if any two distinct vertices of Γ are connected by at least k internally disjoint rainbow paths. The *rainbow-k-connectivity* of Γ, denoted by $\mathrm{rc}_k(\Gamma)$, is the minimum number of colors required to color the edges of Γ to make it rainbow-k-connected. We usually denote $\mathrm{rc}_1(\Gamma)$ by $\mathrm{rc}(\Gamma)$, which is called the *rainbow connection number* of Γ.

In [5] and [6], Chartrand et al. first introduced the concept of rainbow k-connectivity for $k = 1$ and $k \geq 2$, respectively. Rainbow k-connectivity has application in transferring information of high security in communication networks. For details we refer to [6] and [8]. The NP-hardness of determining $\mathrm{rc}(\Gamma)$ was shown by Chakraborty et al. [4]. Recently, the rainbow connectivity of some special classes of graphs have been studied; see [12] for complete graphs, [11] for regular complete bipartite graphs, [10, 14, 15] for Cayley graphs and [16] for power graphs. For more information, see [13].

For a non-abelian group G, the *non-commuting graph* Γ_G of G has the vertex set $G \setminus Z(G)$ and two distinct vertices x and y are adjacent if $xy \neq yx$, where $Z(G)$

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is the center of G. According to [17] non-commuting graphs were first considered by Erdős in 1975. Over the past decade, non-commuting graphs have received considerable attention. For example, Abdollahi et al. [1] proved that the diameter of any non-commuting graph is 2. For two non-abelian groups with isomorphic non-commuting graphs, the sufficient conditions that guarantee their orders are equal were provided by Abdollahi and Shahverdi [2] and Darafsheh [7]. Akbari and Moghaddamfar [3] studied strongly regular non-commuting graphs. Solomon and Woldar [18] characterized some simple groups by their non-commuting graphs.

In this paper we study the rainbow k-connectivity of non-commuting graphs and obtain the following results.

Theorem 1.1 *Let G be a finite non-abelian group. Then $\mathrm{rc}_2(\Gamma_G) = 2$. In particular, $\mathrm{rc}(\Gamma_G) = 2$.*

Theorem 1.2 *For any positive integer k, there exist infinitely many non-abelian groups G such that $\mathrm{rc}_k(\Gamma_G) = 2$.*

2 Preliminaries

In this section we present some lemmas which we need in the sequel.

For vertices x, y of a graph Γ, let $\tau(x, y)$ be the number of the common neighbors of x and y.

Lemma 2.1 *Let G be a finite non-abelian group, and let x and y be two distinct vertices of Γ_G. Then $\tau(x, y) \geq \frac{1}{6}|G|$.*

Proof. For each $g \in G$, $C_G(g)$ denotes the centralizer of g in G. By the principle of inclusion and exclusion,

$$\begin{aligned}\tau(x, y) &= |G| - |C_G(x) \cup C_G(y)| \\ &= |G| - |C_G(x)| - |C_G(y)| + |C_G(x) \cap C_G(y)| \\ &\geq |G| - |C_G(x)| - |C_G(y)| + \frac{|C_G(x)| \cdot |C_G(y)|}{|G|}.\end{aligned}$$

If $|C_G(x)| = |C_G(y)| = \frac{1}{2}|G|$, then

$$\tau(x, y) \geq \frac{|C_G(x)| \cdot |C_G(y)|}{|G|} = \frac{1}{4}|G|;$$

if not,

$$\tau(x, y) \geq |G| - |C_G(x)| - |C_G(y)| \geq \frac{1}{6}|G|. \qquad \square$$

The lexicographic product $\Gamma \circ \Lambda$ of graphs Γ and Λ has the vertex set $V(\Gamma) \times V(\Lambda)$, and two vertices $(\gamma, \lambda), (\gamma', \lambda')$ are adjacent if $\{\gamma, \gamma'\} \in E(\Gamma)$, or if $\gamma = \gamma'$ and $\{\lambda, \lambda'\} \in E(\Lambda)$.

Lemma 2.2 *Let G be a non-abelian group and A be an abelian group of order n. Then*

$$\Gamma_{G\times A} \cong \Gamma_G \circ \overline{K_n},$$

where $\overline{K_n}$ is the complement of the complete graph K_n.

For positive integers l, r and t, let $K_{l[r]}$ denote a complete l-partite graph with each part of order r, and let $K_{l[r],t}$ denote a complete $(l+1)$-partite graph with l parts of order r and a part of order t.

Lemma 2.3 *Let D_{2n} and Q_{4m} be respectively the dihedral group of order $2n$ and the generalized quaternion group of order $4m$, where $n \geq 3$ and $m \geq 2$. Then*

(i) *If n is odd, then $\Gamma_{D_{2n}} \cong K_{n[1],n-1}$.*

(ii) *If n is even, then $\Gamma_{D_{2n}} \cong K_{\frac{n}{2}[2],n-2}$.*

(iii) $\Gamma_{Q_{4m}} \cong K_{m[2],2m-2}$.

Li and Sun [12] studied the rainbow k-connectivity of some families of complete multipartite graphs. Now we compute the rainbow k-connectivity of another family.

Proposition 2.4 *Let $m \geq n+1$, $lmn \neq 2$. Then $\mathrm{rc}_2(K_{m[l],ln}) = 2$.*

Proof. Write $\Gamma = K_{m[l],ln}$. Let $\{a_{j,i} : 1 \leq j \leq l\}$ and $\{a_{j,m+1} : 1 \leq j \leq ln\}$ be all parts of Γ, where $i = 1, \ldots, m$.

Case 1. $n = 1$.

Case 1.1. $m = 2$.

If $l = 2r$, then we assign a color to the edges

$$\begin{aligned}&\{a_{2j-1,1}, a_{2j-1,2}\}, \{a_{2j-1,1}, a_{2j,2}\}, \{a_{2j-1,1}, a_{2j-1,3}\}, \{a_{2j,1}, a_{2j-1,2}\},\\ &\{a_{2j,1}, a_{2j,2}\}, \{a_{2j,1}, a_{2j,3}\}, \{a_{2j-1,2}, a_{2j-1,3}\}, \{a_{2j-1,2}, a_{2j,3}\}, \quad 1 \leq j \leq r\end{aligned} \tag{1}$$

and another color to the remaining edges.

If $l = 2r+1$, then we assign a color to the edges

$$\begin{aligned}&\{a_{l,2}, a_{l,3}\}, \{a_{l,1}, a_{2j-1,2}\}, \{a_{l,1}, a_{2j,3}\}, \{a_{l,2}, a_{2j-1,1}\}, \{a_{l,2}, a_{2j,1}\},\\ &\{a_{l,2}, a_{2j-1,3}\}, \{a_{l,2}, a_{2j,3}\}, \{a_{l,3}, a_{2j,2}\}, \quad 1 \leq j \leq r\end{aligned}$$

and the edges in (1), and another color to all other edges.

Case 1.2. $m = 3$.

The edges

$$\{a_{j,1}, a_{j,2}\}, \{a_{j,2}, a_{j,4}\}, \{a_{j,3}, a_{j,4}\}, \quad 1 \leq j \leq l$$

are assigned by a color and all other edges are assigned by another color.

Case 1.3. $m \geq 4$.

The edges

$$\{a_{j,i}, a_{j,i+1}\}, \{a_{j,m+1}, a_{j,1}\}, \quad 1 \leq i \leq m, \quad 1 \leq j \leq l$$

are assigned by a color and all other edges are assigned by another color.

Note that all the above colorings make Γ rainbow-2-connected. Hence $\mathrm{rc}_2(\Gamma) = 2$.

Case 2. $n \geq 2$.

We assign a color to

$$\{a_{j,i}, a_{j,i+1}\}, \{a_{j,1}, a_{j,m}\}, \{a_{j,i'}, a_{(j-1)n+i',m+1}\}, 1 \leq i \leq m-1, 1 \leq i' \leq n, 1 \leq j \leq l$$

and another color to the remaining edges. Note that this coloring makes Γ rainbow-2-connected. This implies that $\mathrm{rc}_2(\Gamma) = 2$. $\square$

3 Proof of main results

In this section, we shall prove Theorems 1.1 and 1.2.

Proposition 3.1 *Let G be a finite non-abelian group with $|G| \geq 114$. Then $\mathrm{rc}_2(\Gamma_G) = 2$.*

Proof. We randomly color the edges of Γ_G with two colors. Denote by $\mathcal{P}_G$ the probability that such a random coloring makes it not rainbow-2-connected. It suffices to prove that $\mathcal{P}_G < 1$.

Let x and y be two distinct vertices of Γ_G. If x and y are adjacent, then the probability that there exist no rainbow paths of length 2 from x to y is $(1/2)^{\tau(x,y)}$. If x and y are non-adjacent, then the probability that Γ_G has precisely a rainbow path of length 2 from x to y is $\tau(x,y)(1/2)^{\tau(x,y)}$, and the probability that Γ_G has no rainbow paths of length 2 from x to y is $(1/2)^{\tau(x,y)}$. Note that

$$|E(\Gamma_G)| = \frac{1}{2} \sum_{x \in V(\Gamma_G)} (|G| - |C_G(x)|) \geq \frac{1}{4}|G|(|G| - |Z(G)|). \tag{2}$$

Write

$$\mathcal{P} = \sum_{x \sim y} \left(\frac{1}{2}\right)^{\tau(x,y)} + \sum_{x \nsim y} \left(\left(\frac{1}{2}\right)^{\tau(x,y)} + \tau(x,y) \left(\frac{1}{2}\right)^{\tau(x,y)} \right), \tag{3}$$

where $x \sim y$ denotes that x and y are adjacent. Now

$$\begin{aligned}
\mathcal{P}_G &\leq \mathcal{P} \\
&\leq \sum_{x\sim y}\left(\frac{1}{2}\right)^{\frac{1}{6}|G|}+\sum_{x\nsim y}\left(\frac{1}{2}\right)^{\frac{1}{6}|G|}+\sum_{x\nsim y}|G|\left(\frac{1}{2}\right)^{\frac{1}{6}|G|} \qquad \text{(by Lemma 2.1)} \\
&= \binom{|V(\Gamma_G)|}{2}\left(\frac{1}{2}\right)^{\frac{1}{6}|G|}+\sum_{x\nsim y}|G|\left(\frac{1}{2}\right)^{\frac{1}{6}|G|} \\
&\leq \binom{|G|-|Z(G)|}{2}\left(\frac{1}{2}\right)^{\frac{1}{6}|G|} \\
&\quad +|G|\left(\frac{1}{2}\right)^{\frac{1}{6}|G|}\left(\binom{|G|-|Z(G)|}{2}-\frac{1}{4}(|G|-|Z(G)|)|G|\right) \qquad \text{(by (2))} \\
&= \frac{1}{4}(|G|-|Z(G)|)(|G|^2-2|Z(G)|-|Z(G)||G|-2)\left(\frac{1}{2}\right)^{\frac{1}{6}|G|} \\
&< \left(\frac{1}{2}\right)^{\frac{1}{6}|G|+2}|G|^3.
\end{aligned}$$

It follows that if $|G| \geq 114$, then $\mathcal{P}_G < 1$. □

Proposition 3.2 *Let G be a finite non-abelian group with $|G| < 114$. Then $\mathrm{rc}_2(\Gamma_G) = 2$.*

Proof. Let $\mathcal{P}_G$ be the probability that such a random coloring makes Γ_G not rainbow-2-connected. Thus if $\mathcal{P}_G \leq 1$, then $\mathrm{rc}_2(\Gamma_G) = 2$. Using GAP [9], we compute $\mathcal{P}$ (see (3)) by the following code.

```
M:=Elements(G);
k:=0;
s:=0;
for i in [1..Size(M)] do
 if Centralizer(G,M[i])<>G then
  for j in [1..Size(M)] do
   if Centralizer(G,M[j])<>G and IsAbelian(Group(M[i],M[j]))=false
                                                  and M[i]<>M[j] then
               t:=Order(G)-Size(Union(Elements(Centralizer(G,M[i])),
                                     Elements(Centralizer(G,M[j]))));
               k:=k+(1/2)^t;
   fi;
  od;
 fi;
od;
for i in [1..Size(M)] do
 if Centralizer(G,M[i])<>G then
  for j in [1..Size(M)] do
```

```
   if Centralizer(G,M[j])<>G and IsAbelian(Group(M[i],M[j]))
                                           and M[i]<>M[j] then
       t:=Order(G)-Size(Union(Elements(Centralizer(G,M[i])),
                             Elements(Centralizer(G,M[j]))));
       s:=s+(1/2)^t+t*(1/2)^t;
   fi;
  od;
 fi;
od;
P:=(s+k)/2;
```

By this code, one gets that $\mathcal{P}_G \leq \mathcal{P} < 1$ with the following exceptions:

(i) $D_6, D_8, Q_8, D_{10}, D_{12}, Q_{12}, D_{14}, D_6 \times \mathbb{Z}_3, D_8 \times \mathbb{Z}_3, Q_8 \times \mathbb{Z}_3$, and all non-abelian groups of order 16.

(ii) G_1 and G_2, where $G_1 = $ SmallGroup(32,49) and $G_2 = $ SmallGroup(32,50) in GAP.

Note that $\Gamma_H \cong K_{4[2],6}$ or $K_{3[4]}$ for any non-abelian group H of order 16. By Lemmas 2.2, 2.3 and Proposition 2.4, the rainbow 2-connectivity of the non-commuting graph of each group in (i) is 2.

Next we shall prove that $\mathrm{rc}_2(\Gamma_{G_1}) = \mathrm{rc}_2(\Gamma_{G_2}) = 2$. Note that $\Gamma_{G_1} \cong J(6,2) \circ \overline{K_2}$. For convenience, in the following we use ab to denote the set $\{a, b\}$ for two distinct letters a, b. Write $\Gamma = J(6,2) \circ \overline{K_2}$ and

$$V(\Gamma) = \{a_i a_j, b_i b_j : 1 \leq i, j \leq 6, i \neq j\},$$

$$E(\Gamma) = \{\{a_i a_j, a_i a_k\}, \{b_i b_j, b_i b_k\}, \{a_i a_j, b_i b_k\} : 1 \leq i, j, k \leq 6, i \neq j, i \neq k, j \neq k\}.$$

We assign the edges in $\{\{a_i a_j, a_i a_k\} : i > \max\{j, k\}\}$, $\{\{b_i b_j, b_i b_k\} : i > \max\{j, k\}\}$ and $\{\{a_i a_j, b_i b_k\} : i < \min\{j, k\}\}$ a color and all other edges another color. It follows that $\mathrm{rc}_2(\Gamma) = 2$. Hence $\mathrm{rc}_2(\Gamma_{G_1}) = \mathrm{rc}_2(\Gamma_{G_2}) = 2$. □

Combining Propositions 3.1 and 3.2, we complete the proof of Theorem 1.1.

Proof of Theorem 1.2: By Theorem 1.1, we may assume $k \geq 3$. Note that for any non-abelian group H, $\kappa(\Gamma_H)$ is divisible by $|Z(H)|$ by [1, Proposition 2.4]. Therefore, we may choose a non-abelian group G with $k \leq \kappa(\Gamma_G)$. Next we prove that $\mathrm{rc}_k(\Gamma_G) = 2$ if $|G|$ is large enough.

We randomly color the edges of Γ_G with two colors. Denote by $\mathcal{P}_G$ the probability that such a random coloring makes it not rainbow-2-connected. It suffices to show that $\mathcal{P}_G < 1$. Let x and y be distinct vertices of Γ_G. If x and y are adjacent, then the probability that there exist no k rainbow paths of length 2 from x to y is

$$\sum_{i=0}^{k-2} \binom{\tau(x,y)}{i} (1/2)^{\tau(x,y)}.$$

If x is not adjacent to y, then the probability that there are no k rainbow paths of length 2 from x to y is

$$\sum_{i=0}^{k-1} \binom{\tau(x,y)}{i} (1/2)^{\tau(x,y)}.$$

Write $|G| = n$. Then we have

$$\begin{aligned}
\mathcal{P}_G &\leq \sum_{x\sim y}\sum_{i=0}^{k-2}\binom{\tau(x,y)}{i}\left(\frac{1}{2}\right)^{\tau(x,y)} + \sum_{x\nsim y}\sum_{i=0}^{k-1}\binom{\tau(x,y)}{i}\left(\frac{1}{2}\right)^{\tau(x,y)} \\
&\leq \sum_{x\sim y}\sum_{i=0}^{k-2}\tau(x,y)^i\left(\frac{1}{2}\right)^{\tau(x,y)} + \sum_{x\nsim y}\sum_{i=0}^{k-1}\tau(x,y)^i\left(\frac{1}{2}\right)^{\tau(x,y)} \\
&\leq \left(\frac{1}{2}\right)^{\frac{1}{6}n}\left(\sum_{x\sim y}\sum_{i=0}^{k-2}n^i + \sum_{x\nsim y}\sum_{i=0}^{k-1}n^i\right) \qquad \text{(by Lemma 2.1)} \\
&= \left(\frac{1}{2}\right)^{\frac{1}{6}n}\left(\sum_{i=0}^{k-2}\binom{|V(\Gamma_G)|}{2}n^i + \left(\binom{|V(\Gamma_G)|}{2} - |E(\Gamma_G)|\right)n^{k-1}\right) \\
&< \left(\frac{1}{2}\right)^{\frac{1}{6}n}\left(\sum_{i=0}^{k-2}n^{i+2} + n^{k+1}\right) \\
&= \frac{\sum_{i=2}^{k+1}n^i}{2^{\frac{n}{6}}}.
\end{aligned}$$

This implies that $\mathcal{P}_G < 1$ if n is large enough. □

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